Encoding and decoding of additional logic in the phase space of all operators

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Abstract This report describes an approach for encoding and decoding of discrete information about basic states at the input of operators in the phase of their outputs. The decoding is considered as a special case of interference with four different forms of decoding that reflect the classes of identity and negation operators. If an operator decode another, he is able to read the information encoded in the phase space, and reduce the encoded bits to state or its negation. Decoding relationships have been developed both as regards the parameters of the operator and in terms of Boolean functions encoding. This further leads to an increase in the level of abstraction. The approach of the proposed system differs from previous discussions of phase encoding, making encoding a substantial part of all operators so that the correct encoded information can be determined from the parameters of the operators.

Keywords Quantum operators · Phase space · Quantum circuit · Error correction code

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1. INTRODUCTION

This research is focused on a summary of the primitive, elementary quantum operators and therefore is closely linked with the work, carried out by Barenco, etc., on elementary circuits and establishing universality of single qubit and two qubit operators [1, 2]. This research differs from previous works on the primitive quantum operators in that its goal is not universality, but systematization. Although the possibility for constructing as much as possible different circuits from a small number of operators is important, the focus here is shifted on the possibility of the operator to express another logic, operating in the circuit, and abstract quantum operators in respect of this logic. Given that the single qubit operators are a major component of the standard set of universal operators for quantum calculations, then they are the primary focus of this research. The developed by the author formalized qubit operator [6, 7, 8] presented in this research uses the decomposition of the phase/amplitude, which characterizes the operators, presented in this report. The formalized operator allows operation with single qubit operators as linear combinations of the operator for identity and the operator for negation. In such a case, a single qubit operator either negates, identifies, or performs partial identity / negation. The relative weights of each basis operator are presented in relation to the probability distribution, which they extract from the single qubit basic states, unlike the more common probability amplitude. Something more, the different phases of the basis operators are logically formalized, in order to ensure a unitary operator. This formalization of the phase space shows a clear separation between the operators, formed by intensional basis, and the operators, formed by non-intensional basis. To increase the abstraction, the composition of the basis operators is developed entirely in terms of the operator parameters.

The formalization of operators is used for development of a system for phase encoding and decoding with single qubit operators. In this system the operators encode some function of the basic states, on which they operate, to the phase of their result. Then the decoding is examined as a special case of interference of the operators. If one operator decodes another one, it is able to "read" the information encoded in the phase space, and to decrease the bit to the encoded state or its logic negation. The decoding relationships are developed both in respect of the operator parameters, as well as in respect of the Boolean encoding functions. By looking at the decoding relationships of the encoding functions, it is possible to be determined the exact way in which an interference is generated, and any residual encoding, remaining from the decoding.

For the parameters $a \in [0, 1]$ and s, $t, v \in \mathbb{C}$ c $\lfloor s \rfloor = |t| = |v| = 1$, U parametrizes a single qubit operator such that

$$U\ (a; s; t; v) = \begin{pmatrix} \sqrt{a} & s\sqrt{1-a} \\ t\sqrt{1-a} & v\sqrt{a} \end{pmatrix} \qquad (1.1)$$

It is assumed, that this operator is sufficient to capture single qubit operators, but later can be seen, that the lack of ability for determination of the phase of the upper left point in the matrix reduces the expressiveness of the form. The operator, given in equation 1.2, is designed to maximize particular restriction of the modification in the Grover's search.

$$\begin{pmatrix} \sqrt{1-a} & \sqrt{a} \\ \sqrt{a} & -\sqrt{1-a} \end{pmatrix} \qquad (1.2)$$

In this research the approach to the operators differs from that one of Grover in the fact that it is tried to be developed a structure, which expresses all operators, and not just a subset, related with a certain problem or algorithm. Grover shows this semi-general form in the context of a modified search. Of interest is the finding of a general form, which could capture all single qubit operators. In this work are separated the phase and the amplitude components of the operator. This separation is made in order to be implemented a logic with more classic understandings of probabilities, while highlighting and identifying the ways in which the changes in the phase generate different interference patterns.
The next iteration of the formalized operator is very similar to the final summary, presented in this report. It includes a parameter in the fourth phase and means the phase parameters as a set $\mathbb{P} = \{\phi_0, \phi_1, \ \phi_{2,}, \phi_3\}$, such that $\phi_i = e^{i\psi_i}$ at $0 \le \psi_i < 2\pi$.

$$U\ (a;\ \mathbb{P}) = \begin{pmatrix} \phi_0\sqrt{a} & \phi_1\sqrt{1-a} \\ \phi_2\sqrt{1-a} & \phi_3\sqrt{a} \end{pmatrix} \qquad (1.3)$$

In order to guarantee that $U\ (a,\ \mathbb{P})$ is unitary, it is necessary the operators to belong to one of the two classes, based on their phase parameterization.

1. *Class I:* The phase of the points in one column are equivalent, while the phases of another column are not.
2. *Class II:* For each ϕ_i phase of the adjacent points differs by a factor of $k\pi$ for odd *k*.

The implemented in this work $U\ (a,\mathbb{P})$ parameterization includes a summary of two qubit controlled operators. The summary of the controlled operators uses two formalized single qubit operators, which are conditionally applied to the target qubit, based on the value of the control qubit. At this stage, a simple set of formalized quantum operators has already been established and the focus is shifted on the ways in which they could be used in the design and analysis of quantum algorithms. The perspective that the quantum operators encode information to the phase space at their output, is not new to this work. In fact, this is the main element at the analysis of many quantum algorithms [3, 4]. Similarly to the case with the used from Grover common single qubit operator, this encoding and decoding is treated usually on a suitable basis, and it is not necessary to become an integral part of the result of the operator. The formalized operator can be used for the strict determination of the encoding and decoding behavior of elementary and complex operators. From there the coding and decoding are transformed into a rule; all the operators encode information for states on which they act on the phase of the state, which derives from their application. This encoding is expressed in terms of the Boolean functions, well-established area from the classical calculations.

2. PHASE ENCODING

The single qubit operators extract their encoding functions from the set of the single-byte Boolean functions. *The set* B^1 *is equal to* $\{f | f : \mathbb{B} \mapsto \mathbb{B}\}$ *= {ID, NOT, ZERO, ONE}, where*
$ID: b \to b$
$NOT: b \to \bar{b}$
$ZERO: b \to 0$
$ONE: b \to 1$
Three binary axes can logically formalize the space of the single qubit, basis operators with real values:

- The parameter of the global phase γ determines whether a given gate is a combination of ±I and ±N or ±Z and ±X.
- The parameter of the phase of identity ι determines the phase of the operator for identity.
- The parameter of the phase of negation η determines the phase of the operator for negation.

First will be defined functions, which connect the logical phase parameters with the more structural formalization of the classes for identity and negation *Id* and *Neg.* For the achievement of this objective let's $\gamma = 0$ be the combination of the operators, and $\iota = 0$ and $\eta = 0$ indicate the positive phase respectively for the operators for identity and negation. This leads to the next enumeration, based on the $\gamma\iota\eta$.
$000 \to I + N \quad 100 \to Z + X$
$001 \to I - N \quad 101 \to Z - X$
$010 \to -I + N \quad 110 \to -Z + X$
$011 \to -I - N \quad 111 \to -Z - X$
With this allocation of space, it is possible to be defined functions, which connect the enumeration based on the logical parameters with the parameters of the function *Id* and *Neg.*

Definition 1 *The parameters* (γ, ι) *are connected with* $(\iota, \gamma \oplus \iota\,)$ *through the function* $\mathfrak{I}_{\gamma\eta} = Id_{i(\gamma\, \oplus\, \iota\,)}$*, defined as*
$\mathfrak{I}_{00} = Id_{00} = I$

$\mathfrak{I}_{01} = Id_{11} = -I$
$\mathfrak{I}_{10} = Id_{01} = Z$
$\mathfrak{I}_{11} = Id_{10} = -Z$ (2)

Definition 2 *The parameters (γ,η) are connected with $((\overline{\gamma \oplus \eta}),\eta)$ through the function $\mathcal{N}_{\gamma\eta} = Neg_{(\overline{\gamma\oplus\eta})\eta}$, defined as*
$\mathcal{N}_{00} = Neg_{10} = N$
$\mathcal{N}_{01} = Neg_{01} = -N$
$\mathcal{N}_{10} = Neg_{00} = X$
$\mathcal{N}_{11} = Neg_{11} = -X$ (3)

Both functions provide the possibility for defining the desired connection with the arguments *Id* and *Neg.*
$\gamma\iota\eta = \left(\mathfrak{I}_{\gamma\iota}, Neg_{\gamma\eta}\right)$ (4)

If $\mathcal{U}_1$ is the space of unitary operators of the two-dimensional Hilbert space above $\mathbb{R}$ and $\mathbb{R}_{[0,1]}$ is the interval[0,1]. It is defined $U : \mathbb{R}_{[0,1]} \times \mathbb{B}^3 \to \mathcal{U}_1$ through:
$U(\alpha,\gamma\iota\eta) = \sqrt{\alpha}\mathfrak{I}_{\gamma\iota} + \sqrt{1-\alpha}\mathcal{N}_{\gamma\eta}$ (5)

For $x \in \mathbb{B}$, the action of the formalized operator $U(\alpha,\gamma\iota\eta)$ of $|x\rangle$ is
$U(\alpha,\gamma\iota\eta)|x\rangle = \sqrt{a}\mathfrak{I}_{\gamma\iota}|x\rangle + \sqrt{1-a}\mathcal{N}_{\gamma\iota}|\bar{x}\rangle$ (6)

Since $\mathfrak{I}$ and $\mathcal{N}$ give basis operators, which at most should introduce a phase change to $|x\rangle$ and $|\bar{x}\rangle$ it is possible to be indicated their influences as phase functions of the form $(-1)^{f(x)}$, where $f \in \mathcal{B}^1$.

The phase functions for *I and -I* obviously are respectively *ZERO* and *ONE,* since they apply changes on the global phase. In the case of *Z* and *-Z* it is noted that the presented phase changes are balanced through all input / output pairs, and not through constants.
$Z|0\rangle = |0\rangle$
$Z|1\rangle = -|1\rangle$
$-Z|0\rangle = -|0\rangle$
$-Z|1\rangle = |1\rangle$

From the phase influences, represented by *Z* and *-Z*, it follows that the phase function, corresponding to *Z,* is *ID*, with its dual *NOT, corresponding to -Z.* From here the next connection from $\gamma\iota$ to $\mathcal{B}^1$ determines the phase functions, based on the parameters of the phase:

$$\gamma\iota \to \mathcal{B}^1 \begin{cases} ZERO & \gamma\iota = 00 \\ ONE & \gamma\iota = 01 \\ ID & \gamma\iota = 10 \\ NOT & \gamma\iota = 11 \end{cases} \quad (7)$$

Similarly to the case with $\pm I$, the phase functions for *X* and *-X* are respectively constants *ZERO* and *ONE* due to global phase shifts. A research for the influence of *N* and *–N* of the phase of the result reveals the same balanced model as *Z and -Z*; the positive phase operator corresponds to *ID,* while the negative - corresponds to *NOT.*
$N|0\rangle = |1\rangle$
$N|1\rangle = -|0\rangle$
$-N|0\rangle = -|1\rangle$
$-N|1\rangle = |0\rangle$

This leads to a connection for $\gamma\eta$, which is similar to that one for $\gamma\iota$. In this case, when $\gamma = 0$, the functions are in the set *BAL*, and when $\gamma = 1$, they are in the set *CONST.* This is opposite of the connection for the operators in $\mathfrak{I}$. For $\gamma\eta$

$$\gamma\eta \to \mathcal{B}^1 \begin{cases} ID & \gamma\eta = 00 \\ NOT & \gamma\eta = 01 \\ ZER & \gamma\eta = 10 \\ ONE & \gamma\eta = 11 \end{cases} \quad (8)$$

It is now possible to be defined an encoding function $\mathcal{E}$, connecting $\gamma\iota\eta \in \mathbb{B}^3$ with the pairs of the functions in $\mathcal{B}^1$. Subscripts are used to indicate the specific functions within the pair.

Phase coding functions of single qubit operators

$$\mathcal{E}(\gamma\iota\eta) = \begin{cases} (ZERO, ID) & \gamma\iota\eta = 000 \\ (ZERO, NOT) & \gamma\iota\eta = 001 \\ (ONE, ID) & \gamma\iota\eta = 010 \\ (ONE, NOT) & \gamma\iota\eta = 011 \\ (ID, ZERO) & \gamma\iota\eta = 100 \\ (ID, ONE) & \gamma\iota\eta = 001 \\ (NOT, ZERO) & \gamma\iota\eta = 110 \\ (NOT, ONE) & \gamma\iota\eta = 111 \end{cases} \qquad (9)$$

For $\mathcal{E}(0)$ the expression $\mathcal{E}(0)_0$ designates the function *ZERO,* and $\mathcal{E}(0)_1$ – *ID.* Perhaps the most interesting feature of $\mathcal{E}$ is that it connects $\mathbb{B}^3$ with the operators in $(BAL \times CONST) \cup (CONST \times BAL)$, and not $\mathcal{B}^1 \times \mathcal{B}^1$ generally. With the help of these links can be determined the impact of the formalized operator on the basic state, without the explicitly calling up the basic matrices.

$$U(\alpha, \gamma\iota\eta)\big|x\rangle = -1^{\varepsilon(\gamma\iota\eta)0(x)}\sqrt{a}\big|x\rangle + -1^{\varepsilon(\gamma\iota\eta)1(x)}\sqrt{1-a}|\bar{x}\rangle \qquad (10)$$

Thus the application of an operator $A = U(\alpha, \gamma\iota\eta)$ to basis $|x\rangle$ encodes the function $\mathcal{E}(\gamma\iota\eta)_{\oplus y}(x)$ to the phase of $\langle y|A|x\rangle$. In respect of the exact input-output pairs,

$$\langle x|A|x\rangle = -1^{\varepsilon(\gamma\iota\eta)0(x)}\sqrt{a}$$

$$\langle \bar{x}|A|x\rangle = -1^{\varepsilon(\gamma\iota\eta)1(x)}\sqrt{1-a} \qquad (11)$$

The formula, based on encoding, given in equation 10 leads to a precise idea for the act of phase encoding. First, *the phase encoding is a feature relative to basic states, and not states as a whole.* The state, which is the result of the application of an operator to the basic state, carries in its phase information for the initial basic state. When considering the application of an operator to a superposition of the basic state, the encoding, which is obtained, is a set of information for non-null basic amplitude states available in the initial state.

3. COMPOSITION, INTERFERENCE AND PHASE DECODING

If the phase encoding is the flow of information from the amplitude state to the phase state, then the decoding is the reverse process. For a better understanding what the decoding of information from a phase space represents and how such decoding might occur, it is good to be examined how the phase encoding functions are composed, as their respective operators are composed in a circuit.

a. Interference

If we consider the consistent application of two operators on the basic state $|x\rangle$. The first operator encodes $\mathcal{E}(\gamma\iota\eta)$ into the phase of $|x\rangle$ and $|\bar{x}\rangle$, while creating a superposition of both states, based on the value of α. If $A = U(\alpha_A, (\gamma\iota\eta)_A)$ is a single qubit operator with a basis $x \in \mathbb{B}$.

$$\langle x|A|x\rangle = (-1)^{\mathcal{E}(\gamma\iota\eta_A)0(x)}\sqrt{a_A}$$

$$\langle \bar{x}|A|x\rangle = (-1)^{\mathcal{E}(\gamma\iota\eta_A)1(x)}\sqrt{1-a_A} \qquad (11)$$

The application of the second operator then affects the state, created by the first operator. These interference patterns generate the probability amplitudes $\langle x|BA|x\rangle$ and $\langle \bar{x}|BA|x\rangle$.

1. $\langle x|A|x\rangle$ with $\langle x|B|x\rangle$ and $\langle x|B|\bar{x}\rangle$ produces the probability amplitude $\langle x|BA|x\rangle$
2. $\langle \bar{x}|A|x\rangle$ with $\langle \bar{x}|B|\bar{x}\rangle$ and $\langle \bar{x}|B|x\rangle$ corresponds to the probability amplitude $\langle \bar{x}|BA|x\rangle$.

It is possible to evaluate these interference patterns with respect to the interaction of the encoded information on the basic states, respectively, by *A and B.* If operator

$$B = U(a_B, (\gamma\iota\eta)_B)$$

$$\langle x|B|x\rangle = (-1)^{\mathcal{E}((\gamma\iota\eta)_B)0(x)}\sqrt{a_B}$$

$$\langle \bar{x}|B|x\rangle = (-1)^{\mathcal{E}((\gamma\iota\eta)_B)1(x)}\sqrt{1-a_B}$$

$$\langle x|B|\bar{x}\rangle = (-1)^{\mathcal{E}((\gamma\iota\eta)_B)1(\bar{x})}\sqrt{1-a_B}$$

$$\langle \bar{x}|B|\bar{x}\rangle = (-1)^{\mathcal{E}((\gamma\iota\eta)_B)0(\bar{x})}\sqrt{a_B} \qquad (12)$$

Given the behaviors of the basic levels of *B and A,* listed in equations 11 and 12, it is possible to be established interference patterns of the basic level, which are formed for $B^{\circ}A$.

$$\langle x|BA|x\rangle = \langle x|B|x\rangle\langle x|A|x\rangle + \langle x|B|\bar{x}\rangle\langle \bar{x}|A|x\rangle$$

$$\langle \bar{x}|BA|x\rangle = \langle \bar{x}|B|\bar{x}\rangle\langle \bar{x}|A|x\rangle + \langle x|A|x\rangle\langle \bar{x}|B|x\rangle \qquad (13)$$

In order to be understood the interference in terms of the phase functions, it is useful to first note that expressions like $\langle x|B|x\rangle\,\langle x|A|x\rangle$ take on the form $(-1)^{f \oplus g}ab$, where f and g are functions in $\mathcal{B}^1$ and

$0 \leq a, b \leq 1$. When these forms are added, the form of the interference is determined by the properties of the phase functions. Now consider the different interference interactions of B with A.

$$\langle x|B|x\rangle\langle x|A|x\rangle| = -1^{\varepsilon((\gamma\iota\eta)B)_0(x)\oplus\varepsilon((\gamma\iota\eta)A)_0(x)}\sqrt{a_B}\sqrt{a_A}|$$
$$\langle x|B|\bar{x}\rangle\langle\bar{x}|A|x\rangle| = -1^{\varepsilon((\gamma\iota\eta)B)_1(\bar{x})\oplus\varepsilon((\gamma\iota\eta)A)_1(x)}\sqrt{1-a_A}\sqrt{1-a_B}|$$
$$\langle\bar{x}|B|\bar{x}\rangle\langle\bar{x}|A|x\rangle| = -1^{\varepsilon((\gamma\iota\eta)B)_0(\bar{x})\oplus\varepsilon((\gamma\iota\eta)A)_1(x)}\sqrt{1-a_A}\sqrt{a_B}|$$
$$\langle x|B|x\rangle\langle\bar{x}|A|x\rangle| = -1^{\varepsilon((\gamma\iota\eta)A)_0(\bar{x})\oplus\varepsilon((\gamma\iota\eta)B)_1(x)}\sqrt{1-a_B}\sqrt{a_A}| \qquad (14)$$

Interference occurs between the two operator and is defined relative to a basis.

Definition 3 *If A and B are single qubit operators. The operator B* constructively interferes *with A, if* $|\langle x|BA|x\rangle| > |\langle x|A|x\rangle|$.

Definition 4 *If A and B are single qubit operators. The operator B* destructively interferes *with A, if* $|\langle x|BA|x\rangle| < |\langle x|A|x\rangle|$.

It is obvious that when dealing with a single qubit, if the probability amplitude relative to a basis increases, then the other probability amplitude must be decreased. In other words, when a constructive interference occurs relative to some basis $|x\rangle$, then there should be also a destructive interference relative to $|\bar{x}\rangle$. In the n qubit space it must be true that when one probability amplitude is increased, at least one another must be decreased. Is not mandatory the decreasing amplitude to be that one of the logical negation of the state, whose probability amplitude has increased. Generally speaking, most of the operators will create some form of interference.

b. Decoding

Phase decoding is the name given to a set of well-defined interference patterns. For example, the action of the operator A can be seen as encoding identifying information for the state $|x\rangle$ during the phase of $A|x\rangle$. If operator B is a decoder of A, then it will effectively read that the encoded information acts on the amplitudes of $A|x\rangle$. Logically, a single qubit decoder might decode information in the phase space in one of two ways.

1. Identifies the encoded state in such way that $BA|x\rangle$ and $|x\rangle$ are at least extensionally equivalent. Thus, the combined effect of BA is like an operator in Ext_1.
2. Negates the encoded state in such way that $BA|x\rangle$ is at least extensionally negation of $|x\rangle$. Thus the combined effect of BA is like an operator in $Next_1$. The decoding, as well as all forms of interference, can be examined in relation to the understanding for the interaction and relationship between the encoding, performed by the operators in question.

Identity decoders

Identity decoder B of operator A introduces an interference pattern such that a previously encoded state is created, with a phase change. This form of interference can be understood in terms of the composition of operators. In the space of the single qubit operators extensional identity decoder acts together with the decoded operator, to create either the basis operators *Z*, or - *Z*. Generally speaking the extensional identity decoders might introduce phase changes to an arbitrary basis. In other words, they can decode one basic state such that $BA|x\rangle = |x\rangle$, and another - $BA|y\rangle = -|y\rangle$.

Corollary 1 *If A and B are single qubit, formalized operators. Then BA* is an identity decoder *only if*
$a_A = a_B$ и $\gamma_B \oplus (\iota_A \oplus \eta_B) \oplus (\iota_B \oplus \eta_B) = 1$

Proof. First, it should be noted that, the product *BA might be simplified*

$$BA = \left(\sqrt{a_B}\mathfrak{I}_{(\gamma\iota)_B} + \sqrt{1-a_B}\mathcal{N}_{(\gamma\eta)_B}\right)\left(\sqrt{a_A}\mathfrak{I}_{(\gamma\iota)_A} + \sqrt{1-a_A}\mathcal{N}_{(\gamma\eta)_A}\right)$$

$$\sqrt{a_B a_A}\mathfrak{I}_{(\gamma_B\oplus\gamma_A)(\iota_B\oplus\iota_A)} + \sqrt{a_B(1-a_A)}\mathcal{N}_{(\gamma_B\oplus\gamma_A)(\gamma_B\oplus\iota_B\oplus\eta_A)}$$
$$+\sqrt{(1-a_A)a_A}\mathcal{N}_{(\gamma_B\oplus\gamma_A)(\eta_B\oplus\iota_A)} + \sqrt{(1-a_B)(1-a_A)}\mathfrak{I}_{(\gamma_B\oplus\gamma_A)(\overline{\gamma_B}\oplus\eta_A\oplus\eta_B)}$$

Furthermore, the following are equivalent,

$$\gamma_B\,(\iota_A \oplus \eta_B) \oplus (\iota_B \oplus \eta_B) = 1$$

$$\overline{\gamma_B \oplus \eta_B \oplus \eta_A} = (\iota_A \oplus \iota_B)$$

$$\gamma_B \oplus \iota_B \oplus \eta_A = (\overline{\iota_A \oplus \eta_B})$$

All operators for identity and negation have the same global phase $\gamma_A \oplus \gamma_B$. In order for the above to be true, the operators for negation must be neutralized, i.e. to have different phases, and the operators for identity must be combined, i.e. to have the same phase. Thus

$\iota_B \oplus \iota_A = \overline{\gamma_B} \oplus \eta_A \oplus \eta_B$ и $\gamma_B \oplus \iota_B \oplus \eta_A \neq \eta_B \oplus i_A$

These two corollaries are equivalent to each other and to

$\gamma_B \oplus (\iota_A \oplus \eta_B) \oplus (\iota_B \oplus \eta_B) = 1$

And, finally, taking into account the restrictions on the amplitude parameters, it is clear that $\alpha_A = \alpha_B$. Thus, when

$BA \in \mathfrak{I}_{yx}$ follows that $\alpha_A = \alpha_B$ and $\gamma_B \oplus (\iota_A \oplus \eta_B) \oplus (\iota_B \oplus \eta_B) = 1$

Formal consequence 1 *If operator B is identity decoder of A, then*

$BA = \mathfrak{I}_{(\gamma_A \oplus \gamma_B)(I_A \oplus I_B)}$

Proof. Follows from Corollary 1.

By this general formula it is possible to be determined a clearer picture of the decoding and the relationship between the encoded and decoded states. Ultimately, the understanding of the conditions, under which the identity decoding occurs, leads to a means for connecting an operator to its identity decoders and vice versa.

Corollary 2 *If operator B =* $U(\alpha_B, (\gamma\iota\eta)_B)$ *is identity decoder of A =* $U(\alpha_A, (\gamma\iota\eta)_A)$*. Then, if* $\alpha_A = \alpha_B$

$$BA = \begin{cases} I & \gamma_B = \gamma_A = 0, i_A = \eta_B, \eta_A \neq i_B \text{ или } \gamma_B = \gamma_A = 1, i_A = \eta_B, \eta_A = i_B \\ -I & \gamma_B = \gamma_A = 0, i_A \neq \eta_B, \eta_A = i_B \text{ или } \gamma_B = \gamma_A = 1, i_A \neq \eta_B, \eta_A \neq i_B \\ Z & \gamma_A = 0, \gamma_B = 1, i_A = \eta_B, \eta_A = i_B \text{ или } \gamma_A = 1, \gamma_B = 0, i_A = \eta_B, \eta_A \neq i_B \\ -Z & \gamma_A = 0, \gamma_B = 1, i_A \neq \eta_B, \eta_A \neq i_B \text{ или } \gamma_A = 1, \gamma_B = 0, i_A \neq \eta_B, \eta_A = i_B \end{cases}$$

(3.2.13)

Proof. Follows from Corollary 1 and consequence 1.

Corollary 2 allows to be defined specifically the phases of the identity decoders for a given operator.

	I	-I	Z	-Z
$(\gamma\iota\eta)_A$	$\alpha_A = \alpha_B$			
000	*001*	*010*	*100*	*111*
001	*000*	*011*	*101*	*110*
010	*011*	*000*	*110*	*101*
011	*010*	*001*	*111*	*100*
100	*100*	*110*	*001*	*010*
101	*101*	*111*	*000*	*011*
110	*110*	*100*	*011*	*000*
111	*111*	*101*	*010*	*001*

Table 1: The identity decoding by phase number γιη

Table 1 shows the conditions for identity decoding with respect to the operator parameters. In all cases, the amplitude of the decoder must match that one of the operator, which it decodes. Each row of the table corresponds to the operator, which is being decoded, and the value in the first column is that one of its phase. The remaining columns give the desired phase for each decoder by the result of the combined action of decoder and encoder. For example, if $A = U(\alpha, 101)$ should be decoded identically, then row 101 lists the operators, which will carry out this decoding. More precisely, it can be seen that $U(\alpha, 000)^\circ U(\alpha, 101) = Z$ since phase 000 is listed as a Z decoder for phase 101.

Negation decoders

The development of the negation decoders more or less follows the line of the identity decoders. The negation decoders create states, which are extensional negations of the encoded information. The determination of the exact formalization of the negation decoders for single qubit operators is really simple.

Corollary 3 *If the operators A and B are single qubit, formalized operators. Then BA is the negation decoder* only *if* $\alpha_A = 1 - \alpha_B$ *and* $\gamma_B \oplus (\iota_A \oplus \eta_B) \oplus (\iota_B \oplus \eta_A) = 0$.

Proof. First, the product *BA* can be simplified,

$BA = \left(\sqrt{\alpha_B}\mathfrak{I}_{(yi)_B} + \sqrt{1-\alpha_B}\mathcal{N}_{(y\eta)_B}\right)\left(\sqrt{\alpha_A}\mathfrak{I}_{(yi)_A} + \sqrt{1-\alpha_A}\mathcal{N}_{(y\eta)_A}\right)$

$= \sqrt{\alpha_B\alpha_A}\mathfrak{I}_{(y_B \oplus y_A)\oplus(\iota_B \oplus l_A)} + \sqrt{\alpha_B(1-\alpha_A)}\mathcal{N}_{(y_B \oplus y_A)\oplus(y_B \oplus \iota_B \oplus \eta_A)} +$

$\sqrt{(1-\alpha_B)\alpha_A}\mathcal{N}_{(y_B \oplus y_A)\oplus(\eta_B \oplus l_A)} + \sqrt{(1-\alpha_B)(1-\alpha_A)}\mathfrak{I}_{(y_A \oplus y_B)\oplus(\overline{y_B} \oplus \eta_A \oplus \eta_B)}$

The following are equivalent,
$and\, \gamma_B \oplus (\iota_A \oplus \eta_B) \oplus (\iota_B \oplus \eta_A) = 0$
$\gamma_B \oplus \iota_B \oplus \eta_A = (\eta_A \oplus \iota_B)$
$\gamma_B \oplus \iota_A \neq (\overline{\gamma_B} \oplus \eta_A \oplus \eta_B)$

If it is assumed that
$\alpha_A = (1 - \alpha_B)$ и $and\, \gamma_B \oplus (\iota_A \oplus \eta_B) \oplus (\iota_B \oplus \eta_A) = 0$
then

$$BA = \sqrt{(1-a_A)a_A}\mathfrak{T}_{(\gamma_B \oplus \gamma_A)(\gamma_B \oplus \eta_A \oplus \eta_B)} + (1-a_A)\mathcal{N}_{(\gamma_B \oplus \gamma_A)(\eta_A \oplus i_B)} +$$
$$a_A \mathcal{N}_{(\gamma_B \oplus \gamma_A)(\eta_B \oplus i_A)} + \sqrt{a_A(1-a_A)}\mathfrak{T}_{(\gamma_A \oplus \gamma_B)(\overline{\gamma_B} \oplus \eta_A \oplus \eta_B)}$$
$$= \mathcal{N}_{(\gamma_B \oplus \gamma_A)(\eta_B \oplus i_A)}$$
$$= \mathcal{N}_{(\gamma_B \oplus \gamma_A)(\gamma_B \oplus i_B \oplus \eta_A)} \qquad (15)$$

Then it must be true that $BA = N_{yx}$ for some $yx \in \{0,1\}^2$.
All operators for identity and negation have the same global phase $\gamma_A \oplus \gamma_B$. In order for the above to be true, the operators for identity must be neutralized, i.e. to have different phases, and the operators for negation must be combined, i.e. to have the same phase. Thus
$\gamma_B \oplus \iota_A \neq (\overline{\gamma_B} \oplus \eta_A \oplus \eta_B)$ и $\gamma_B \oplus \iota_B \oplus \eta_A = (\eta_A \oplus \iota_B)$
These two corollaries are equivalent to each other and to
$\gamma_B \oplus (\iota_A \oplus \eta_B) \oplus (\iota_B \oplus \eta_A) = 0$
And, finally, taking into account the restrictions on the amplitude parameters, it is clear that $\alpha_A = 1 - \alpha_B$. Thus, when $BA \in \mathcal{N}_{yx}$
follows that $\alpha_A = 1 - \alpha_B$ and
$\gamma_B \oplus (\iota_A \oplus \eta_B) \oplus (\iota_B \oplus \eta_A) = 0$
This leads to a new common formula for the negation decoders.

Formal consequence 2 *If operator B is negation decoder of A. Then*
$BA = \mathcal{N}_{(\gamma_B \oplus \gamma_A)(\eta_B \oplus I_A)}$
Proof. Follows from equation 15 in Corollary 3.

Corollary 4 *If operator B =* $U(\alpha_B, (\gamma\iota\eta)_B)$ *negation decoder of A =* $U(\alpha_A, (\gamma\iota\eta)_A)$*. Then, if* $\alpha_B = 1 - \alpha_A$.

$$BA = \begin{cases} N & \gamma_B = \gamma_A = 0, i_A = \eta_B, \eta_A = i_B \text{ или } \gamma_B = \gamma_A = 1, i_A = \eta_B, \eta_A \neq i_B \\ -N & \gamma_B = \gamma_A = 0, i_A \neq \eta_B, \eta_A \neq i_B \text{ или } \gamma_B = \gamma_A = 1, i_A \neq \eta_B, \eta_A = i_B \\ X & \gamma_A = 0, \gamma_B = 1, i_A = \eta_B, \eta_A \neq i_B \text{ или } \gamma_A = 1, \gamma_B = 0, i_A = \eta_B, \eta_A = i_B \\ -X & \gamma_A = 0, \gamma_B = 1, i_A \neq \eta_B, \eta_A = i_B \text{ или } \gamma_A = 1, \gamma_B = 0, i_A \neq \eta_B, \eta_A \neq i_B \end{cases} \qquad (16)$$

Proof. Follows from Corollary 2 and its consequence 2.
The decoding connections may be outlined in terms of the phase numbers, as shown in table 2.

	N	-N	X	-X
$(\gamma\iota\eta)_A$	$\alpha_A = 1 - \alpha_B$			
000	*000*	*011*	*110*	*101*
001	*010*	*001*	*100*	*111*
010	*001*	*010*	*111*	*100*
011	*011*	*000*	*101*	*110*
100	*110*	*101*	*000*	*011*
101	*100*	*111*	*010*	*001*
110	*111*	*100*	*001*	*010*
111	*101*	*110*	*011*	*000*

Table 2: The negation decoding by phase number γιη

c. A new look at decoding

Until now the decoding was limited to the conditions under which the two operators are combined in order to form a certain basis operator. The requirements for the parameters are shown in table 3

	I	Z	N	-X
$(\gamma\iota\eta)_A$	$\alpha_A = \alpha_B$		$\alpha_A = 1 - \alpha_B$	
000	*001*	*100*	*000*	*110*
001	*000*	*101*	*010*	*100*

010	*011*	*110*	*001*	*111*
011	*010*	*111*	*011*	*101*
100	*100*	*001*	*110*	*000*
101	*101*	*000*	*100*	*010*
110	*110*	*011*	*111*	*001*
111	*111*	*010*	*101*	*011*

Table 3 The decoding operators

More common structure of the decoding operators occurs when calling the main matrix formulation of the operators.

Corollary 5 *If operator $U \in \{\pm I, \pm Z, \pm X, \pm N\}$ and A is any single qubit, formalized operator. If $U \in \{\pm I, \pm Z\}$, then $UA^\dagger$ is an identity decoder of A. If $U \in \{\pm N, \pm X\}$, then $UA^\dagger$ is a negation decoder of A.*
Proof. For all single qubit operators, A, $A^\dagger A = I$. Therefore $(UA^\dagger)A = U$. If $U \in \mathrm{Ext}_1$, then $(UA^\dagger)A \in \mathrm{Ext}_1$ and thus $UA^\dagger$ is an identity decoder of A. If $U \in N\mathrm{ext}_1$, then $(UA^\dagger)A \in N\mathrm{ext}_1$ and thus $UA^\dagger$ is a negation decoder of A. Corollary 5 summarizes the single qubit decoders and allows the reconstruction of the previous results. Furthermore, it provides the internal necessity for development of a unified system for consideration of decoding operators for operators of any size.

	I	Z	N	-X
$\mathcal{E}(\gamma\iota\eta)_A$	$\alpha_A = \alpha_B$		$\alpha_A = 1 - \alpha_B$	
(ZERO,ID)	(ZERO,NOT)	(ID,ZERO)	(ZERO,ID)	(NOT,ZERO)
(ZERO,NOT)	(ZERO,ID)	(ID,ONE)	(ONE,ID)	(ID,ZERO)
(ID,ZERO)	(ID,ZERO)	(ZERO,NOT)	(NOT,ZERO)	(ZERO,ID)
(ID,ONE)	(ID,ONE)	(ZERO,ID)	(ID,ZERO)	(ONE,ID)

Table 4 Decoding by encoding function $\mathcal{E}$

In Table 4 is condensed the decoding table with phases 000, 001, 100 and 101, and are reformulated the operators with respect to their encoding functions. The remaining operators are simply negations of those four and their encoding functions. The functions, which decode them, can be defined by reversing the functions for their negation to the complementary function.
To be consider the decoding with respect to the interaction between encoding functions, it is important to be know where and how these encoding functions interact. If at first is reviewed the interference, generated between the operators A and B.

$$\langle x|AB|x\rangle = (-1)^{\varepsilon((\gamma\iota\eta)_B)_0(x)\oplus\varepsilon((\gamma\iota\eta)_A)_0(x)}\sqrt{a_A a_B} + (-1)^{\varepsilon((\gamma\iota\eta)_B)_1(\bar{x})\oplus\varepsilon((\gamma\iota\eta)_A)_1(x_{tB})}\sqrt{(1-a_A)(1-a_B)}$$
$$\langle \bar{x}|AB|x\rangle = (-1)^{\varepsilon((\gamma\iota\eta)_B)_0(\bar{x})\oplus\varepsilon((\gamma\iota\eta)_A)_1(x)}\sqrt{(1-a_A)a_B} + (-1)^{\varepsilon((\gamma\iota\eta)_B)_1(x)\oplus\varepsilon((\gamma\iota\eta)_A)_0(x_{tB})}\sqrt{a_A(1-a_B)}$$

When B is a decoder of A, the amplitudes of the two members in both sums equivalent and the decoding is solely dependent on the relationship between the functions
$\varepsilon((\gamma\iota\eta)_B)_0 \oplus \varepsilon((\gamma\iota\eta)_A)_0$ и $(\varepsilon((\gamma\iota\eta)_B)_1 \circ NOT) \oplus \varepsilon((\gamma\iota\eta)_A)_1$
$(\varepsilon((\gamma\iota\eta)_B)_0 \circ NOT) \oplus \varepsilon((\gamma\iota\eta)_A)_1$ и $\varepsilon((\gamma\iota\eta)_B)_1 \oplus \varepsilon((\gamma\iota\eta)_A)_0$
When an identical decoding occurs, then the encoding functions are complementary in $|x\rangle$ and are neutralized in $|\bar{x}\rangle$. The opposite is true for the negation decoding. Informatively may be examined, how these interference patterns are combined. When operator B decodes operator A, then their encoding functions combine to determine the phase / sign of the members, given above. Four of these functions can be represented as a matrix C

$$C = \begin{pmatrix} \varepsilon((\gamma\iota\eta)_B)_0 \oplus \varepsilon((\gamma\iota\eta)_A)_0 & (\varepsilon((\gamma\iota\eta)_B)_1 \circ NOT) \oplus \varepsilon((\gamma\iota\eta)_A)_1 \\ (\varepsilon((\gamma\iota\eta)_B)_0 \circ NOT) \oplus \varepsilon((\gamma\iota\eta)_A)_1 & \varepsilon((\gamma\iota\eta)_B)_1 \oplus \varepsilon((\gamma\iota\eta)_A)_0 \end{pmatrix}$$

where each function determines the phase of the members

$$\begin{pmatrix} \langle x|B|x\rangle\langle x|A|x\rangle & \langle x|B|\bar{x}\rangle\langle \bar{x}|A|x\rangle \\ \langle \bar{x}|B|\bar{x}\rangle\langle \bar{x}|A|x\rangle & \langle \bar{x}|B|x\rangle\langle x|A|x\rangle \end{pmatrix}$$

	I	Z
(ZERO,ID)	$\begin{pmatrix} \mathrm{ZERO} & \mathrm{ZERO} \\ ID & NOT \end{pmatrix}$	$\begin{pmatrix} \mathrm{ID} & \mathrm{ID} \\ ONE & ZERO \end{pmatrix}$
(ZERO,NOT)	$\begin{pmatrix} \mathrm{ZERO} & \mathrm{ZERO} \\ NOT & ID \end{pmatrix}$	$\begin{pmatrix} \mathrm{ID} & \mathrm{ID} \\ ZERO & ONE \end{pmatrix}$
(ID,ZERO)	$\begin{pmatrix} \mathrm{ZERO} & \mathrm{ZERO} \\ NOT & ID \end{pmatrix}$	$\begin{pmatrix} \mathrm{ID} & \mathrm{ID} \\ ZERO & ONE \end{pmatrix}$

(ID,ONE)	$\begin{pmatrix} \text{ZERO} & \text{ZERO} \\ ID & NOT \end{pmatrix}$	$\begin{pmatrix} \text{ID} & \text{ID} \\ ONE & ZERO \end{pmatrix}$

Table 5: Identity decoding patterns

It can be easily checked that the decoding patterns for the negations of the basis operators is an complement of the patterns, given in Tables 5 and 6, i.e. the matrix, in which all operators have been replaced by their complement in $\mathcal{B}^1$.

	N	X
(ZERO,ID)	$\begin{pmatrix} \text{ZERO} & \text{ONE} \\ ID & ID \end{pmatrix}$	$\begin{pmatrix} \text{NOT} & \text{ID} \\ ZERO & ZERO \end{pmatrix}$
(ZERO,NOT)	$\begin{pmatrix} \text{ONE} & \text{ZERO} \\ ID & ID \end{pmatrix}$	$\begin{pmatrix} \text{ID} & \text{NOT} \\ ZERO & ZERO \end{pmatrix}$
(ID,ZERO)	$\begin{pmatrix} \text{ONE} & \text{ZERO} \\ ID & ID \end{pmatrix}$	$\begin{pmatrix} \text{ID} & \text{NOT} \\ ZERO & ZERO \end{pmatrix}$
(ID,ONE)	$\begin{pmatrix} \text{ZERO} & \text{ONE} \\ ID & ID \end{pmatrix}$	$\begin{pmatrix} \text{NOT} & \text{ID} \\ ZERO & ZERO \end{pmatrix}$

Table 6: Models of negation decoding

The decoding patterns for the negations of the four operators, given above, are the same as their corresponding negations. For example, the pattern for the operator with a phase 011 is the same as that one of 000. The decoding patterns, given in Tables 5 and 6, allow it to be casted a glance, oriented to the encoding / decoding, to the interaction between the operators, as they are applied sequentially. Of particular importance is the result, which *the residual encoding,* left by the decoding, has on the subsequent operators. The residual encoding is the encoding / decoding presentation of the operator, to which is limited the combination of a decoder and the operator, which it decodes. For example, decoding to *Z and N* leaves the encoding *ID(x) on* the decoded states $|x\rangle$ and $|\bar{x}\rangle$, as these two operators encode the function *ID.* The overall effect of the encoding in the presence of a residual encoding can be handled, by first reducing the decoder and the operator for encoding to a suitable basis operator and then combining it with the next operator. This also highlights that residual encoding characterizes the result of applying an arbitrary non-basis operator after a basis operator. The basis operator would perform the same phase encoding, then the next operator encodes with respect to the encoding of the basis.
If $A = U(\alpha, \gamma\iota\eta)$, then

$$A\big((-1)^{f(x)}|x\rangle\big) = (-1)^{f(x)}A|x\rangle$$
$$= (-1)^{f(x)\oplus(\gamma\iota\eta)_0(x)}\sqrt{a}|x\rangle + (-1)^{f(x)\oplus(\gamma\iota\eta)_1(x)}\sqrt{(1-a)}|\bar{x}\rangle \qquad (17)$$

As seen in equation 17, operator A effectively performs the encoding of an operator with encoding function $(\varepsilon(\gamma\iota\eta)_0 \oplus f, \varepsilon(\gamma\iota\eta)_1 \oplus f)$

When the function *f is* the residual of a negation decoding, it should be considered that *f* is relative to negation of the encoded state. In other words, the residual functions, given in table 6 are not *f* from equation 17. If the residual function of the negation decoding is *g,* the resulting state is $(-1)^{g(x)}|\bar{x}\rangle = (-1)^{(g\,\circ\, NOT)(\bar{x})}|\bar{x}\rangle$ and through equation 17 it is visible that operator *A* performs the encoding
$(\varepsilon(\gamma\iota\eta)_0 \oplus (g \circ NOT)), \varepsilon(\gamma\iota\eta)_1 \oplus (g \circ NOT)$

4. ENCODING AND DECODING AT $\alpha \in \{0, 1\}$

The links between the encoding and decoding, discussed in the previous sections, have been developed with an eye to the operators with amplitude parameters, which are not strictly zero or non-null. When the amplitude parameter is zero (respectively one), it is fair to say that $\mathcal{E}_1$, respectively $\mathcal{E}_0$, is encoded to the negation (respectively the identity) of the input data. The zero amplitude of such state effectively "neutralizes" the encoding. Essentially, it would be wrong to be presented a state with zero amplitude as bearing any encoded information in its phase, as it actually has no phase. When are considered the operators themselves, the difference is not substantial. When assessing the information encoded to a certain state by a series of operators, it is important to be taken into account the lack of a phase change, which occurs at the application of operators with amplitude parameter 0 or 1.
The function: $\varepsilon' : \mathbb{R}_{[0,1]} \times \mathbb{B}^3 \to \mathcal{B}^1 \times \mathcal{B}^1$ provides a pair of encoding functions, which are aware of the amplitude.
More precisely, $\varepsilon'(a, \gamma\iota\eta) = E_{a,\gamma\iota\eta} = (E_0^{a,\gamma\iota\eta}, E_1^{a,\gamma\iota\eta})$

Where

$$E_0^{a,\gamma\iota\eta} = \begin{cases} \varepsilon(\gamma\iota\eta)_0 & 0 \leq a \leq 1 \\ ZERO & a = 1 \end{cases} \qquad E_1^{a,\gamma\iota\eta} = \begin{cases} \varepsilon(\gamma\iota\eta)_1 & 0 \leq a \leq 1 \\ ZERO & a = 0 \end{cases} \tag{18}$$

5. CONCLUSIONS

This report describes how can be used formalized logic for phase encoding and decoding, in order to formalize the idea of encoding discrete information on the basic input states in the output phase of the operator. The approach of this system differs from previous discussions of phase encoding, making the encoding a substantial part of all operators, so that the correct encoded information can be determined from the parameters of the operators.

The decoding is considered as a special case of interference with four different forms of decoding that reflect the classes of identity and negation operators discussed in the previous chapters. The identity decoders determine the state encoded in the phase space and reduce the output state to the encoded state. Corollary 1 gives the requirements of the parameters for identity decoding. Its corollary 1 allows a reduction of the composition of an operator with its identical decoder to a basis operator only with regard to the operator parameters. The negation decoders negate the encoded state, leaving the state in the negation of the previous encoded state. Therefore, all forms of decoding can be strictly managed in terms of the operator parameters. This further leads to an increase in the level of abstraction. Decoding is also examined from the perspective of Boolean functions that perform encoding. The proposed system provides the means for addressing the decoding links with regard to the encoding functions; it also allows the determination of any residual encoded information after the decoding has happened.

The proposed system provides the means for addressing the decoding links with regard to the encoding functions; it also allows the determination of any residual encoded information after the decoding has happened. In addition, it addresses, in terms of the Boolean encoding functions, the precise nature of the interference, which leads to a phase decoding. Then is researched the effect which the residual encoding has on the following operators. By developing, a perspective focused on the Boolean functions, for the phase space is built a level of abstraction beyond the "negative probabilities", allowing the interference to be addressed in terms of primitive Boolean functions.

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